Example (More details for Remark 2 in Handout 1): Let $0 < a < 1$, define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \begin{cases} ax + x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$. Then, $f'(x) = \begin{cases} a + 2x \sin \frac{1}{x} - \cos \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$, where we use the definition of the derivative to compute $f'(0)$, and $f''(x) = \begin{cases} (2 - \frac{1}{x}) \sin \frac{1}{x} - \frac{2}{x} \cos \frac{1}{x} & \text{if } x \neq 0 \\ \text{does not exist} & \text{if } x = 0 \end{cases}$.

Claim: There is no solution to the following system of equations

$$(1) \qquad \begin{cases} f'(x) & = 0 \\ f''(x) & = 0 \end{cases} \iff \begin{pmatrix} 2x & -1 \\ 2 - \frac{1}{x^2} & -\frac{2}{x} \end{pmatrix} \begin{pmatrix} \sin \frac{1}{x} \\ \cos \frac{1}{x} \end{pmatrix} = \begin{pmatrix} -a \\ 0 \end{pmatrix}$$

We prove this claim by contradiction. Suppose that system (1) has a solution for some x, then observe that, for this x in the system (1), the following system

$$(2) \qquad \begin{pmatrix} 2x & -1 \\ 2 - \frac{1}{x^2} & -\frac{2}{x} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -a \\ 0 \end{pmatrix}$$

has a solution $x_1 = \sin \frac{1}{x}$, $x_2 = \cos \frac{1}{x}$. (3)

However, if we solve the system (2) by using Cramer's rule, we shall get

$$x_1 = \frac{\begin{vmatrix} -a & -1 \\ 0 & -\frac{2}{x} \end{vmatrix}}{\begin{vmatrix} 2x & -1 \\ 2 - \frac{1}{x^2} & -\frac{2}{x} \end{vmatrix}} = \frac{2a/x}{-4 + (2 - \frac{1}{x^2})} \quad ; \quad x_2 = \frac{\begin{vmatrix} 2x & -a \\ 2 - \frac{1}{x^2} & 0 \end{vmatrix}}{\begin{vmatrix} 2x & -1 \\ 2 - \frac{1}{x^2} & -\frac{2}{x} \end{vmatrix}} = \frac{2a - a/x^2}{-4 + (2 - \frac{1}{x^2})}$$

which implies that $x_1^2 + x_2^2 = \dfrac{a^2[4x^2 + (2x^2 - 1)^2]}{(1 + 2x^2)^2} = \dfrac{a^2[4x^4 + 1]}{1 + 4x^2 + 4x^4} \leq a^2 < 1$ (4)

This is a contradiction since, by (3), $x_1 = \sin \frac{1}{x}$, $x_2 = \cos \frac{1}{x}$ is a solution of the system (2), and satisfies that $x_1^2 + x_2^2 = \sin^2 \frac{1}{x} + \cos^2 \frac{1}{x} = 1 > a^2 = x_1^2 + x_2^2$ by (4). Therefore, the system (1) does have a solution. This implies that when $f'(x) = 0$, $f''(x) \neq 0$, therefore, f is not $1 - 1$ around x. Note that the Inverse Function Theorem does not apply here since f' is not continuous at $x = 0$.

Examples of curves in parametric form:

(1) Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined by $f(t) = (t, |t|)$. Note that the f is not differentiable at $t = 0$, and $f(\mathbb{R}) =$ the graph of $y = |x|$ over $\mathbb{R}$.

(2) Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined by $f(t) = (t^3, t^2)$. Note that the f is everywhere differentiable, but $f'(0) = (0, 0)$, and $f(\mathbb{R}) =$ the graph of $y^3 = x^2$ over $\mathbb{R}$ with a vertical cusp at $(0, 0)$, i.e. $\lim_{x \to 0^\pm} \frac{dy}{dx} = \pm\infty$.

(3) Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined by $f(t) = (t^3 - 4t, t^2 - 4)$. Note that f is differentiable and $f'(t) \neq (0, 0)$ for all $t \in \mathbb{R}$. But, $f(2) = f(-2) = (0, 0)$ i.e. f is not a globally one-to-one function, and the curve $f(\mathbb{R})$ is not differentiable at $(0, 0)$.

Example 4(b) in Handout 1: Let $\mathscr{F} = \{f_n(x) = x^n \mid n \in \mathbb{N}\ x \in I = [0, 1]\}$. Note that for each $n \in \mathbb{N}$, and any $x, y \in I$, there exists a c_n lying between x and y such that

$$(*) \qquad |f_n(x) - f_n(y)| = n\, c_n^{n-1}\, |x - y| \quad \text{(by the Mean Value Theorem)}.$$

For each n, choosing $x = 1$ and $0 \le y < 1$ in $(*)$, we have

$$1 - y^n == n\, c_n^{n-1}\,(1-y)$$

Note that the left-hand-side approaches to 1 as n goes to ∞, i.e. $n\, c_n^{n-1}$ on the right-hand-side approaches to $\dfrac{1}{1-y}$ as n goes to ∞. This implies that if $1 > \epsilon > 0$, then $|f_n(1) - f_n(y)| \ge \epsilon$ for all $0 \le y < 1$, and for all sufficiently large n. Hence, $\mathscr{F}$ is not uniformly equicontinuous.

Some notions you need from before:

Exercise: (1a) Let $S \subset \mathbb{R}^p$. When is S said to be open in $\mathbb{R}^p$? closed in $\mathbb{R}^p$? bounded? compact? What is the limiting (or accumulation, or cluster) points set of S?

Exercise (1b) Let $S \subset \mathbb{R}$, $f : S \to \mathbb{R}^q$. When is f said to be continuous on S? Let $W \subset \mathbb{R}^q$, what is $f^{-1}(W)$?What is $f(S)$? What is the graph of f over S? If S is compact, what can you say about $f(S)$, and the continuity of f on S?

(2a) The Completeness Axiom (Sec. 1.5 in the book): Let $\emptyset \ne S \subset \mathbb{R}$, and let $\mathbb{R}$ be equipped with the Euclidean distance $\|x - y\|$ for any $x,\ y \in \mathbb{R}$ (note that $\|x-y\| = |x-y|$ in $\mathbb{R}$).**If** S **has an upper bound, then**, by "bisecting" the set S, **the least upper bound (or the supremum) of** S, denoted $\sup S = \sup\{x \mid x \in S\}$,**exists in** $\mathbb{R}$**, If** S **has a lower bound, then the greatest lower bound** (or the infimum) of S, denoted $\inf S = \inf\{x \mid x \in S\}$, **exists in** $\mathbb{R}$**.** Using this completeness property of $\mathbb{R}$ (or by "bisecting" S), one can show that

(2b) Theorem 1.18**: Every bounded sequence** $\{x_n\}$ **in** $\mathbb{R}$ ($\Longrightarrow \sup\{x_n \mid n \in \mathbb{N}\},\ \inf\{x_n \mid n \in \mathbb{N}\}$ exist) **has a convergent subsequence** ($\{x_{n_k}\}$ converging to $\sup\{x_n\} \in \mathbb{R}$ or $\inf\{x_n\} \in \mathbb{R}$.)

By "bisecting" S, one can be easily generalize Theorem 1.18 to higher dimension $\mathbb{R}^p$,

(2c) Theorem 1.19**: Every bounded sequence** $\{x_n\} \subset \mathbb{R}^p$ **has a convergent subsequence** (that converges to a point in $\mathbb{R}^p$.)

(2d) Theorem 1.20**:** $\{x_n\}$ **is a Cauchy sequence in** $\mathbb{R}^p$ **iff** $\{x_n\}$ **is convergent in** $\mathbb{R}^p$. [Note: You should check that $\{x_n\}$ is a Cauchy sequence in $\mathbb{R}^p \Longrightarrow \{x_n\}$ is bounded in $\mathbb{R}^p$]

Exercise: explain how $(2a)-(2d)$ indicate that $\mathbb{R}^p$, equipped with the Euclidean distance (metric), **has the completeness property if and only if every Cauchy sequence** $\{x_n\}$ **is convergent in** $\mathbb{R}^p$**.** [e.g. Let x_n denote the n digit decimal representation of $\sqrt{2}$, $x_1 = 1$, $x_2 = 1.4$, $x_3 = 1.41, \cdots$. Note that x_n is a Cauchy sequence in $\mathbb{Q}$ with $\lim_{n\to\infty} x_n = \sqrt{2} \notin \mathbb{Q}$, i.e. The Cauchy sequence $\{x_n\}$ converges but it does not converge in $\mathbb{Q}$, so $\mathbb{Q}$ does not have completeness property.]

(3) Let S be a compact subset of $\mathbb{R}^p$, $f : S \to \mathbb{R}^q$ be a continuous map. Then f is uniformly continuous on S, i.e. $\forall \epsilon > 0,\ \exists\, \delta = \delta(\epsilon) > 0,$ such that for any $x, y \in S$ with $\|x - y\| < \delta$ we have $\|f(x) - f(y)\| < \epsilon$. Compare this with f is continuous on S, i.e. f is continuous at each $x \in S$, $\forall \epsilon > 0$, and for each $x \in S$, $\exists\, \delta = \delta(\epsilon, x) > 0,$ such that for any $y \in S$ with $\|y - x\| < \delta$ we have $\|f(y) - f(x)\| < \epsilon$. [e.g. Let $S = (0,1)$, $f(x) = \dfrac{1}{x}$. Then for each $\epsilon > 0$, $x \in (0,1) \Longrightarrow \dfrac{x}{2} > 0$, since $|f(y) - f(x)| = \dfrac{|y-x|}{yx} < \dfrac{2|y-x|}{x^2}$, we may choose $\delta = \min\{\dfrac{x^2\epsilon}{2}, 1\}$, such that if $|y - x| < \delta$ then $|f(x) - f(y)| < \dfrac{2|y-x|}{x^2} < \dfrac{2\delta}{x^2} \le \epsilon$. Note that δ here varies according to x, the continuity of f is Not uniform on (the noncompact set) $(0,1)$.]